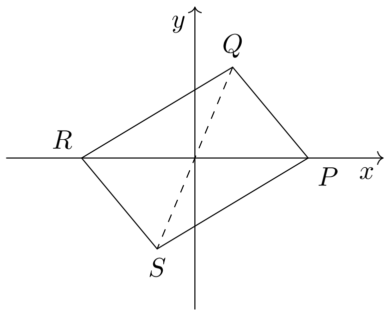
2301. Wlog, we can place the quadrilateral with PR on the x axis and QS passing through the origin:



Since triangles PRQ and PRS have the same area, points Q and S are the same distance from the x axis; therefore, because the diagonal QS passes through the origin, $\overrightarrow{OS} = -\overrightarrow{OQ}$. By symmetry, then, $\overrightarrow{OR} = -\overrightarrow{OP}$. Hence,

$$\overrightarrow{PQ} = -\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OR} - \overrightarrow{OS} = -\overrightarrow{SR}.$$

Therefore, $PQRS$ is a parallelogram. QED.

2302. Rewriting over base two, we have $2^{-2x-1} = 2^{12x}$. Hence, $-2x - 1 = 12x$, giving $x = -\frac{1}{14}$.

2303. We calculate c^2 in two ways. Firstly, by the cosine rule: $c^2 = 2 - 2\cos 2\theta$. Secondly, using the right-angled triangles, $c = 2\sin\theta$ gives $c^2 = 4\sin^2\theta$. Equating these,

$$\begin{aligned} & 2 - 2\cos 2\theta \equiv 4\sin^2\theta \\ \Longrightarrow\ & \cos 2\theta \equiv 1 - 2\sin^2\theta, \text{ as required.} \end{aligned}$$

2304. We multiply the probability of $3, 4, 5, 6, 7$ in that order by the number of orders:

$$5! \cdot \tfrac{4}{52} \cdot \tfrac{4}{51} \cdot \tfrac{4}{50} \cdot \tfrac{4}{49} \cdot \tfrac{4}{48} = 0.000349 \text{ (3sf)}.$$

2305. There is no guarantee that y and z are significantly correlated. The correlation in (x_i, y_i) could be found primarily among points with e.g. i odd, and the correlation in (x_i, z_i) found primarily among points with i even. Then the correlation in (y_i, z_i) would be a lot weaker than the other two. And, if the correlations in (x_i, y_i) and (x_i, z_i) are each only just significant, then it is possible for (y_i, z_i) to fall short of significance.

2306. The derivative is $2x + 2$. So, at point $(a, a^2 + 2a)$, the tangent is $y-(a^2+2a) = (2a+2)(x-a)$, which simplifies to $y = (2a+2)x - a^2$. The other tangent is $y = (2b+2)x - b^2$. Solving simultaneously,

$$\begin{aligned} & (2a+2)x - a^2 = (2b+2)x - b^2 \\ \Longrightarrow\ & (2a - 2b)x = a^2 - b^2 \\ \Longrightarrow\ & 2(a-b)x = (a-b)(a+b) \\ \Longrightarrow\ & x = \tfrac{1}{2}(a+b), \text{ as required.} \end{aligned}$$

2307. The ellipse can be seen as a unit circle stretched in the x and y directions. The scale factors are a and b respectively. Each of these is a stretch in one dimension, giving area scale factor ab. The unit circle has area π, so the area of the ellipse is πab.

2308. The inequality is $-1 \leq 3x - 1 \leq 1$. Its solution is $x \in [0, 2/3]$. We now consider the domain $[0, 1]$ as a possibility space. Since this has unit length, the length of the successful interval converts directly to probability, so $\mathbb{P}(|3x - 1| \leq 1) = 2/3$.

2309. In each case below, each individual sign change is represented as -1. The total effect once these have been repeated/combined is then shown after the equals sign.

(a) $-1 \times -1 = 1$: no sign change.

(b) $(-1)^3 \times (-1)^2 = -1$: sign change.

(c) $-1 \times 1 = -1$: sign change.

(d) $1 \times -1 = -1$: sign change.

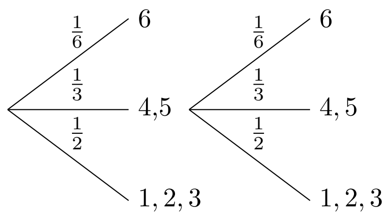
2310. Since $(x + 1)$ is a factor of $x^2 + 2x + 1 \equiv (x + 1)^2$, the lowest common denominator is $x(x + 1)^2$:

$$\begin{aligned} & \frac{1}{x^2 + 2x + 1} + \frac{1}{x} - \frac{1}{x+1} \\ \equiv\ & \frac{x}{x(x+1)^2} + \frac{(x+1)^2}{x(x+1)^2} - \frac{x(x+1)}{x(x+1)^2} \\ \equiv\ & \frac{2x+1}{x(x+1)^2}. \end{aligned}$$

2311. The derivative of $\tan x$ is $\sec^2 x$. By symmetry, then, the derivative of $\cot x$ is $-\operatorname{cosec}^2 x$: we have switched cos for sin, which introduces a minus sign and turns sec into cosec. The result is

$$\int \operatorname{cosec}^2 x\, dx = -\cot x + c.$$

2312. (a) The tree diagram begins



(b) $\mathbb{P}(\text{win within 2}) = \frac{1}{6} + \frac{1}{3} \times \frac{1}{6} = \frac{2}{9}$.

(c) Continuing the calculation started in part (b), $\mathbb{P}(\text{win})$ is given by $\frac{1}{6} + \frac{1}{3} \times \frac{1}{6} + \frac{1}{3}^2 \times \frac{1}{6} + \frac{1}{3}^3 \times \frac{1}{6} + ...$ Written in sigma notation, this is

$$\mathbb{P}(\text{win}) = \sum_{i=1}^{\infty} \tfrac{1}{6} \times \tfrac{1}{3}^{i-1}.$$

(d) This is a geometric series with first term $a = \frac{1}{6}$ and common ratio $r = \frac{1}{3}$. Since $|r| < 1$, its sum to infinity converges to

$$\mathbb{P}(\text{win}) = \frac{\frac{1}{6}}{1 - \frac{1}{3}} = \frac{1}{4}.$$

(e) The game continues until the outcome is one of $\{1, 2, 3, 6\}$. These are equally likely, and one yields a win. Hence, the probability is $\frac{1}{4}$.

2313. The difference between the two regions is the boundary circle $x^2 + y^2 = 1$. A circle (with the word describing a curve, as opposed to that curve's interior) is one-dimensional: it is infinitely thin and has no area.

2314. We can raise both base and input of a logarithm to the same power without changing the output value: $\log_a b \equiv \log_{a^n} b^n$. This yields

(a) $\log_2 x \equiv \log_4 x^2 \equiv 2 \log 4x$,

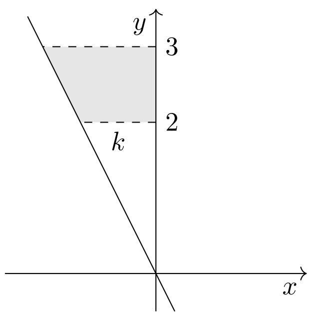
(b) $\log_{16} x \equiv \log_4 \sqrt{x} \equiv \frac{1}{2} \log_4 x$,

(c) $\log_8 x \equiv \log_4 x^{\frac{2}{3}} \equiv \frac{2}{3} \log_4 x$.

2315. All the edges have unit length, so this pyramid is half an octahedron: the vertices A, B, C, D, X are all equidistant from the centre of the base. Hence, the height of X above the base is $\sqrt{2}/2$. The loop then forms similar triangles with each sloping face; since the loop has unit length, the scale factor is 4. So, the height of the loop is

$$h = \frac{3}{4} \times \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{8}.$$

2316. We are a looking for a straight line, through the origin, for which the signed area between the line and the y axis is -6, between $y = 2$ and $y = 3$. So, we need the trapezium below to have area 6:



By similar triangles, the upper side length is $\frac{3}{2}k$, which gives trapezium area $\frac{1}{2}(k + \frac{3}{2}k) = 6$. So, $k = 4.8$. This gives the equation of the line as $x = \frac{-4.8}{2} y$, i.e. $x = -2.4y$.

2317. (a) Setting vertical velocity to zero for the apex, we have $u = 20 \times \frac{3}{5} = 12$, $v = 0$, $a = -10$ and we want t. So, $0 = 12 - 10t$ and $t = 1.2$. At this time the projectile has reached $(16t, 12t - 5t^2)$, which is $(19.2, 10.08)$ m.

(b) Horizontal distance between two successive projectiles is fixed at 16 m. Vertical distance varies, but its minimum value is 0, when the projectiles are at the same height, either side of their respective peaks. We know that this occurs, since $1.2 > 1$: one projectile has not yet reached its peak when the next one is launched. Hence, the closest distance is 16 m.

2318. The compositions are

$$\begin{aligned} \text{fg}(x) &= a(cx + d) + b, \\ \text{gf}(x) &= c(ax + b) + d. \end{aligned}$$

Multiplying out, $acx + ad + b \equiv acx + bc + d$. The x terms are already identical. Equating the constant terms:

$$\begin{aligned} & ad + b = bc + d \\ \Longrightarrow\ & (a-1)d = b(c-1) \\ \Longrightarrow\ & \frac{a-1}{c-1} = \frac{b}{d}. \end{aligned}$$

2319. This can be done by the cosine rule and $\frac{1}{2}ab \sin C$. It is most easily done using Heron's formula:

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where s is the semiperimeter. In this case $a = 3$, $b = 25$, $c = 26$, and $s = 27$. So,

$$A = \sqrt{27 \times 24 \times 2 \times 1} = 36.$$

Hence, $(3, 25, 26)$ is a Heronian triangle.

2320. Assuming the kings can't occupy the same square, there are three cases. With number of threatened squares as $(*)$, the locations for the first king are:

- 4 corners (3),
- 24 sides (5),
- 36 middles (8).

For each, we multiply the probability of the first king being in such a location by the probability that the second king ends up in an adjacent square:

$$\begin{aligned} p &= \tfrac{4}{64} \times \tfrac{3}{63} + \tfrac{24}{64} \times \tfrac{5}{63} + \tfrac{36}{64} \times \tfrac{8}{63} \\ &= \tfrac{5}{48}. \end{aligned}$$

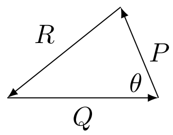
2321. We can ignore the $+k$, since it simply translates the entire problem in the y direction. Hence, the question is whether $y = 0$ is in the range of each reciprocal trig function.

(a) No,

(b) No,

(c) Yes.

2322. The object is in equilibrium, so the sum of the three forces is zero. This gives a triangle of forces:



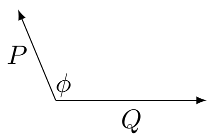
The cosine rule gives

$$\cos\theta = \frac{6^2 + 10^2 - 11^2}{2 \cdot 6 \cdot 10},$$

so $\theta = 82.8°$. The angle between forces **P** and **Q** is therefore $180° - 82.8° = 97.2°$ (1dp).

— Nota Bene —

The answer is 97.2°, as opposed to 82.8°, because vectors, unlike lines, have a sense of $\pm$ associated with their direction. If two *lines* cross at 97.2°, then they also cross at 82.8°, and neither value is better than the other. However, the natural sense of angle for two *vectors* is with both emerging from a single point, as ϕ below:



That way, the angle between two identical vectors is zero, as it should be. Because of the tip-to-tail way in which vectors add, these angles are *exterior* in a triangle of forces.

2323. In each case, we switch input and output, setting the input to y and the output to x.

(a) Rearranging to make y the subject,

$$\begin{aligned} & x = \mathrm{f}(2y-1) \\ \Longrightarrow\ & \mathrm{f}^{-1}(x) = 2y - 1 \\ \Longrightarrow\ & y = \tfrac{1}{2}\big(\mathrm{f}^{-1}(x) + 1\big). \end{aligned}$$

(b) And again,

$$\begin{aligned} & x = \mathrm{fg}(y) \\ \Longrightarrow\ & \mathrm{f}^{-1}(x) = \mathrm{g}(y) \\ \Longrightarrow\ & y = \mathrm{g}^{-1}\mathrm{f}^{-1}(x). \end{aligned}$$

(c) Using g as the inverse of g^{-1},

$$\begin{aligned} & x = \mathrm{g}^{-1}\mathrm{f}(y) \\ \Longrightarrow\ & \mathrm{g}(x) = \mathrm{f}(y) \\ \Longrightarrow\ & y = \mathrm{f}^{-1}\mathrm{g}(x). \end{aligned}$$

2324. (a) Differentiating implicitly,

$$\begin{aligned} & x^2 + xy + y^2 = 1 \\ \Longrightarrow\ & 2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \\ \Longrightarrow\ & \frac{dy}{dx}(x + 2y) = -2x - y \\ \Longrightarrow\ & \frac{dy}{dx} = -\frac{2x+y}{x+2y}. \end{aligned}$$

(b) The x intercepts are at $x^2 = 1$, so $(\pm 1, 0)$. At these points, the gradients are

$$\frac{dy}{dx} = -\frac{\pm 2}{\pm 1} = -2.$$

So, the tangents are $y = -2(x \pm 1)$. These cross the y axis at ± 2, as required.

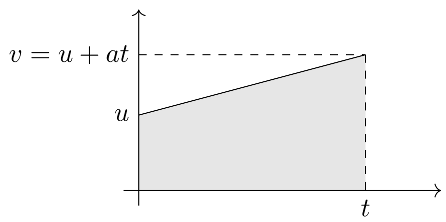
2325. Counting outcomes in the possibility space, we have $(1,6)$ and $(6,1)$ in the former, and $(1,1)$ in the latter. So the difference being five has the greater (doubled) probability.

2326. Equating differences in the AP, $b - a = 2 - b$. Also, equating ratios in the GP, $a/b = 9/a$. These simplify to $b = 1 + \frac{1}{2}a$ and $a^2 = 9b$. Subbing for b,

$$\begin{aligned} & 9 + \tfrac{9}{2}a = a^2 \\ \Longrightarrow\ & a = 6, -\tfrac{3}{2}. \end{aligned}$$

The possible (a, b) pairs are $(6, 4)$ and $(-3/2, 1/4)$.

2327. The velocity-time graph for constant acceleration, with the displacement shaded, is as follows. The acceleration is the gradient, so the final velocity is given by $v = u + at$:



Quoting the formula for the area of a trapezium,

$$\begin{aligned} s &= \tfrac{1}{2}\big(u + (u + at)\big)t \\ &= ut + \tfrac{1}{2}at^2, \text{ as required.} \end{aligned}$$

2328. This is a quadratic in $\sin^{-1} x$, which factorises:

$$\begin{aligned} & (\sin^{-1} x)^2 + \sin^{-1} x = 12 \\ \Longrightarrow\ & (\sin^{-1} x - 3)(\sin^{-1} x + 4) = 0 \\ \Longrightarrow\ & \sin^{-1} x = 3, -4. \end{aligned}$$

The range of the inverse sin function $x \mapsto \sin^{-1} x$ is the domain of the invertible sin function $x \mapsto \sin x$. In radians, this is $[-\pi/2, \pi/2]$. Both 3 and -4 lie outside this interval, so the equation has no roots, as required.

2329. This isn't necessarily true, although it may be. If the alternative hypothesis is $H_1 : p > p_0$, then the critical region consists solely of x values *above* what the null hypothesis would suggest. So, there could be values of k at the lower tail for which $\mathbb{P}(x \le k) < \frac{1}{20}$, but which nevertheless are not in the critical region.

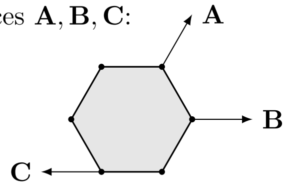
2330. Each of these statements is the negation of one in the factor theorem. The factor theorem says "$\text{f}(x)$ has a factor of $(x-a)$ if and only if $\text{f}(a) = 0$." This implication goes both ways; hence, the implication between the negated statements does too.

2331. The second derivative is zero at $x = 0$, but, for a point of inflection, we also require it to change sign. Algebraically, the second derivative is

$$\frac{d^2y}{dx^2} = 56x^6 + 42x^5 = x^5(56x + 42).$$

The first factor x^5 changes sign at $x = 0$, while the second factor $(56x + 42)$ is positive both sides of $x = 0$. Overall, therefore, we have a sign change: $x = 0$ is a point of inflection.

2332. Call the forces $\mathbf{A}, \mathbf{B}, \mathbf{C}$:



Only force $\mathbf{A}$ has a component in the y direction. Since this is non-zero, the prism must accelerate in the y direction. And only $\mathbf{C}$ has a line of action that doesn't pass through the axis of symmetry of the prism. Since this is non-zero, the prism must also rotate.

2333. Solving for intersections, we get $x^2 + kx = x^3 + kx$, which is $x^2 = x^3$. This is independent of k. The integrand, then, which is the y difference between the curves, is $x^2 + kx - (x^3 + kx) = x^2 - x^3$, which is likewise independent of k. Hence, the definite integral to find the area enclosed is independent of k, as required.

2334. (a) The third Pythagorean trig identity gives

$$\begin{aligned}\cot^2 \tfrac{5\pi}{12} &= \operatorname{cosec}^2 \tfrac{5\pi}{12} - 1 \\ &= \left(\frac{4}{\sqrt{6}+\sqrt{2}}\right)^2 - 1 \\ &= \frac{16}{8 + 4\sqrt{3}} - 1 \\ &= 7 - 4\sqrt{3}.\end{aligned}$$

Reciprocating and square rooting,

$$\tan \tfrac{5\pi}{12} = \frac{1}{\sqrt{7 - 4\sqrt{3}}}.$$

(b) Simplifying the relevant surds,

$$\begin{aligned}\left(2 + \sqrt{3}\right)^2 &= 7 + 4\sqrt{3} \\ \therefore \quad 2 + \sqrt{3} &= \sqrt{7 + 4\sqrt{3}}.\end{aligned}$$

Also, rationalising the denominator from (a),

$$\frac{1}{\sqrt{7 - 4\sqrt{3}}} = \frac{\sqrt{7 + 4\sqrt{3}}}{\sqrt{49 - 48}} = \sqrt{7 + 4\sqrt{3}}.$$

Therefore, $\tan \dfrac{5\pi}{12} = 2 + \sqrt{3}$, as required.

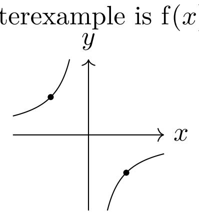
2335. We solve simultaneously by elimination. Adding the equations gives $c + d = 2a$, and subtracting them gives $c - d = 2b$. Dividing each by two,

$$\begin{aligned}a &= \tfrac{1}{2}(c + d), \\ b &= \tfrac{1}{2}(c - d).\end{aligned}$$

2336. The first sentence is correct. However, the second is not. The relevant objects, between which the friction acts, are the runner's feet and the track. If the track were icy, then, in attempting to move in a circle, the runner's feet would slip outwards. So, the "potential motion" in question is that of the runner's feet sliding outwards. Friction acts to oppose this, pushing the runner's feet, and hence the runner, towards the centre of the circle.

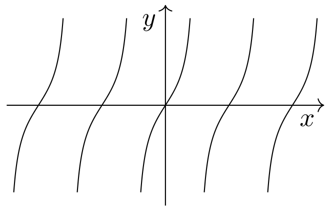
2337. (a) True. A stationary point has $\text{f}'(x) = 0$, but we know that $\text{f}'(x) > 0$ for all x.

(b) False. A counterexample is $\text{f}(x) = -\frac{1}{x}$:



This is increasing everywhere it is defined. However, $1 > -1$ while $\text{f}(1) < \text{f}(-1)$.

(c) False. $\text{f}(x) = \tan(x)$ is increasing everywhere it is defined, and it has infinitely many roots:



2338. We multiply up and equate coefficients, looking for a contradiction:

$$\begin{aligned}&\frac{2x - 1}{x^4 + x^2} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^2 + 1} \\ \Longrightarrow\ &2x - 1 \equiv A(x^3 + x) + B(x^2 + 1) + Cx^2.\end{aligned}$$

The coefficients of x^3 require $A = 0$. Then the LHS has $2x$, whereas the RHS has no terms in x. This is a contradiction. Hence, it is impossible to form such an identity.

2339. Firstly, we write $(uvw)' \equiv ((uv)w)'$. The two-way product rule gives $(uv)'w + (uv)w'$. We use this again on the first term, giving

$$\begin{aligned}&(u'v + uv')w + (uv)w'\\ \equiv\ & u'vw + uv'w + uvw', \text{ as required.}\end{aligned}$$

2340. There are 16 choices for chairperson, leaving 15 people from which to choose two secretaries. This gives $16 \times {}^{15}\mathrm{C}_2 = 1680$.

2341. The curves are circles, radius $\sqrt{k}$, centred on the origin and on (k, k). The centres are $k\sqrt{2}$ apart. Hence, the circles are tangent iff $k\sqrt{2} = 2\sqrt{k}$. Solving, this gives $k = 0$, which is ruled out in the question, or $k = 2$. So, the statement is true.

2342. (a) We can treat the two masses on the slope as a single object. Since these have mass $2m$, the component of their weight acting down the slope is $2mg \sin 30° = mg$. This balances the weight of the hanging mass, so the acceleration of the system is zero. Hence, the tension in the shorter string is $\frac{1}{2}mg$ N.

(b) With friction, the acceleration will still be zero.

i. Roughness in the pulley has no effect on the shorter string, as there is no change in the behaviour of the higher of the two slope masses. The tension is the same as in (a).

ii. With a frictional force acting up the slope, the tension in the shorter string is no longer the only force holding the lower mass in equilibrium. The tension could be lower than in part (a).

2343. This is a GP, first term $a = 1000$ and common ratio $r = \frac{2}{3}$. So, its ordinal formula is $A_n = 1000 \times \frac{2}{3}^{n-1}$. Substituting into the boundary equation,

$$\begin{aligned}&1000 \times \tfrac{2}{3}^{k-1} - 1000 \times \tfrac{2}{3}^{k} = 1\\ \implies & 1000 \times \left(1 - \tfrac{2}{3}\right) = \tfrac{3}{2}^{k-1}\\ \implies & \log_{\frac{3}{2}} \tfrac{1000}{3} = k - 1\\ \implies & k = 15.32...\end{aligned}$$

Hence, the first integer is $k = 16$.

2344. The compound-angle formulae give

$$\begin{aligned}&0 = 2\sin x \cos y - 2\cos x \sin y\\ &\qquad + 2\sin x \cos y + 2\cos x \sin y + 1\\ \implies & 0 = 4\sin x \cos y + 1.\end{aligned}$$

We now have simultaneous equations in $\sin x$ and $\cos y$. Substituting the second into the first,

$$0 = -4\sin^2 x + 1 \implies \sin x = \pm\tfrac{1}{2}.$$

Substituting these values back in, $\sin x = \pm\frac{1}{2}$ and $\cos y = \mp\frac{1}{2}$.

2345. Using the chain rule, $2x + 2y\frac{dy}{dx} = 0$. Making the derivative the subject,

$$\frac{dy}{dx} = -\frac{x}{y}.$$

We can interpret this by by taking the negative reciprocal, which is y/x: this is the gradient of the radius to a point (x, y) on the unit circle. The tangent is perpendicular to the radius.

2346. Multiplying up by the denominators,

$$\begin{aligned}&\frac{1}{2^x + 1} + \frac{1}{2^x - 1} = \frac{1}{4}\\ \implies & 4(2^x - 1) + 4(2^x + 1) = (2^x + 1)(2^x - 1)\\ \implies & (2^x)^2 - 8(2^x) - 1 = 0.\end{aligned}$$

This is a quadratic in 2^x, for which the formula gives

$$2^x = 4 \pm \sqrt{17}$$

We reject the negative root, as $2^x > 0$. So, the solution is

$$x = \log_2\left(4 + \sqrt{17}\right).$$

2347. Let P be the midpoint of AB, and so on. Then the midpoints have position vectors

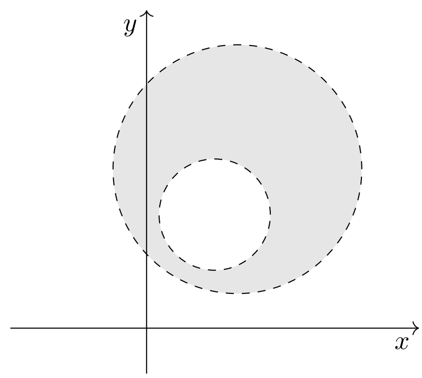
$$\begin{aligned}\mathbf{p} &= \tfrac{1}{2}(\mathbf{a} + \mathbf{b}) & \mathbf{q} &= \tfrac{1}{2}(\mathbf{b} + \mathbf{c})\\ \mathbf{r} &= \tfrac{1}{2}(\mathbf{c} + \mathbf{d}) & \mathbf{s} &= \tfrac{1}{2}(\mathbf{d} + \mathbf{a})\end{aligned}$$

The vectors between midpoints are

$$\begin{aligned}\overrightarrow{PQ} &= \tfrac{1}{2}(\mathbf{b} + \mathbf{c}) - \tfrac{1}{2}(\mathbf{a} + \mathbf{b}) = \tfrac{1}{2}(\mathbf{c} - \mathbf{a})\\ \overrightarrow{SR} &= \tfrac{1}{2}(\mathbf{c} + \mathbf{d}) - \tfrac{1}{2}(\mathbf{d} + \mathbf{a}) = \tfrac{1}{2}(\mathbf{c} - \mathbf{a}).\end{aligned}$$

Since $\overrightarrow{PQ} = \overrightarrow{SR}$, $PQRS$ is a parallelogram. QED.

2348. (a) The boundary equations are circles: centre $(4, 7)$ radius $\sqrt{30}$ and centre $(3, 5)$ radius $\sqrt{6}$. The distance between centres is $\sqrt{1^2 + 2^2}$, which is $\sqrt{5}$. But since $\sqrt{30} - \sqrt{6} > \sqrt{5}$, the smaller circle lies entirely within the larger:



(b) Region R is shaded above. As the difference between circles, it has area $30\pi - 6\pi = 24\pi$.

2349. The derivatives are

$$\frac{dy}{dx} = \mathrm{f}'(x),$$

and, by the chain rule,

$$\frac{dy}{dx} = \frac{\mathrm{f}'(x)}{\mathrm{f}(x)+1}.$$

Substituting a root $x = \alpha$ into the latter and using the fact that $\mathrm{f}(\alpha) = 0$, we get $\frac{dy}{dx} = \mathrm{f}'(\alpha)$. Hence, the two graphs have the same gradient.

Furthermore, they are both at coordinates $(\alpha, 0)$. This is obvious for the first graph. For the second $\mathrm{f}(\alpha) + 1$ has value 1, and $\ln 1 = 0$.

Hence, the graphs are tangent, as required.

2350. The graph is a parabola. So, by symmetry, the vertex must lie midway between the roots, which are a and b. So, the x coordinate of the vertex is $\frac{1}{2}(a+b)$. Substituting this in, we get

$$\begin{aligned} y &= \left(\tfrac{1}{2}(a+b) - a\right)\left(\tfrac{1}{2}(a+b) - b\right) \\ &\equiv \left(\tfrac{1}{2}b - \tfrac{1}{2}a\right)\left(\tfrac{1}{2}a - \tfrac{1}{2}b\right) \\ &\equiv -\tfrac{1}{4}(a-b)^2. \end{aligned}$$

The vertex lies at $\left(\frac{1}{2}(a+b), -\frac{1}{4}(a-b)^2\right)$.

2351. Using $\binom{n}{r} \equiv {}^n\mathrm{C}_r = \dfrac{n!}{r!(n-r)!}$, the LHS is

$$\begin{aligned} &\binom{n-1}{r} - \binom{n-1}{r-1} \\ \equiv\ &\frac{(n-1)!}{r!(n-1-r)!} - \frac{(n-1)!}{(r-1)!(n-r)!}. \end{aligned}$$

We multiply top and bottom of the first fraction by $(n-r)$ and top and bottom of the second fraction by r. This gives

$$\begin{aligned} &\frac{(n-1)!(n-r)}{r!(n-r)!} - \frac{(n-1)!r}{r!(n-r)!} \\ \equiv\ &\frac{(n-1)!(n-r) - (n-1)!r}{r!(n-r)!} \\ \equiv\ &(n-2r)\frac{(n-1)!}{r!(n-r)!}. \end{aligned}$$

Multiplying by $\frac{n}{n}$, this is

$$\begin{aligned} &\frac{n-2r}{n}\frac{n!}{r!(n-r)!} \\ \equiv\ &\frac{n-2r}{n}\binom{n}{r}, \text{ as required.} \end{aligned}$$

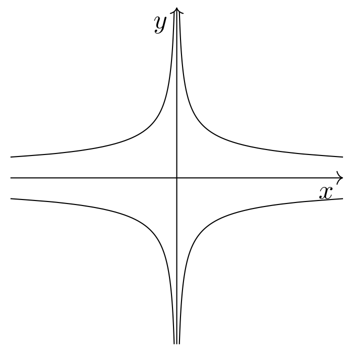
2352. The second Pythagorean trig identity is

$$1 + \tan^2 x \equiv \sec^2 x.$$

Since $\sec^2 x$ and $\tan^2 x$ differ by only a constant, their derivatives are equal, as required.

2353. The forces add to zero: $\mathbf{F}+\mathbf{G}+\mathbf{H} = 0$. Removing e.g. $\mathbf{F}$ gives resultant force $\mathbf{G}+\mathbf{H}$, which is equal to $-\mathbf{F}$. Since the magnitude of the resultant force after removal is equal to that of the force removed, removing the largest force will result in the largest magnitude of acceleration.

2354. Taking the square root, $xy^2 = \pm 1$. This gives two copies of $x = 1/y^2$, with one reflected in the y axis:



2355. (a) Carrying out the integrals, we can combine the constants of integration into a single c on the RHS:

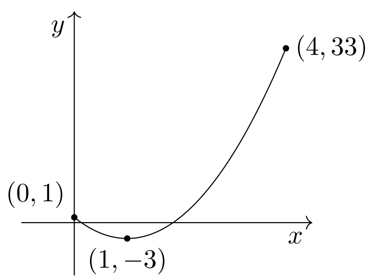
$$\begin{aligned} \int 2y - 1\,dy &= \int \frac{1}{x^2}\,dx \\ \Longrightarrow y^2 - y &= -\frac{1}{x} + c \\ \Longrightarrow y^2 - y + \frac{1}{x} &= c. \end{aligned}$$

(b) We can take the limit, setting $y = 1$ and $\frac{1}{x} = 0$. This gives $c = 0$.

(c) We now have $y^2 - y + \frac{1}{x} = 0$, a quadratic in y. The formula gives

$$\begin{aligned} y &= \frac{1 \pm \sqrt{1 - 4\frac{1}{x}}}{2} \\ \Longrightarrow 2y &= 1 \pm \sqrt{1 - \tfrac{4}{x}}, \text{ as required.} \end{aligned}$$

2356. The derivative is $\frac{dy}{dx} = 8x - 8$, so the vertex of $y = \mathrm{f}(x)$ is at $(1, -3)$, which is in the domain. The endpoints are $(0, 1)$ and $(4, 33)$:



Hence, the range is $\{y \in \mathbb{R} : -3 \le y \le 33\}$.

2357. The curves are a positive quartic below a negative quadratic. The y difference is given by

$$\begin{aligned}\mathrm{h}(x) &= 70 - 20x^2 - (10x^4 + 40x^2)\\ &\equiv 70 - 60x^2 - 10x^4.\end{aligned}$$

The intersections are at the roots of $\mathrm{h}(x)$, which is a quadratic in x^2. It has solution $x = \pm 1$. So, the area enclosed is

$$\begin{aligned}&\int_{-1}^{1} 70 - 60x^2 - 10x^4\,dx\\ &= \Big[70x - 20x^3 - 2x^5\Big]_{-1}^{1}\\ &= (48) - (-48)\\ &= 96.\end{aligned}$$

2358. Integrating, the velocity is $v = -\frac{2}{\pi}\sin(\pi t) + c$. This is a translated sine wave whose average value, over $t \in [0, \infty)$, is c. Therefore, $c = 5$. Integrating again,

$$s = \frac{2}{\pi^2}\cos(\pi t) + 5t + d.$$

Displacement is measured from initial position. So, we substitute $t = 0$ and $s = 0$, giving $d = -\frac{2}{\pi^2}$. Taking out the common factor yields the required result:

$$s = 5t + \frac{2}{\pi^2}(\cos(\pi t) - 1).$$

2359. The unit circle is $x^2 + y^2 = 1$. For intersections, therefore, we require $x^2 + \left(\frac{1}{2}x + k\right)^2 = 1$, which we simplify to $\frac{5}{4}x^2 + xk + k^2 - 1 = 0$. For tangents, this has precisely one root, so $\Delta = 0$:

$$\begin{aligned}&k^2 - 4\cdot\tfrac{5}{4}\cdot(k^2 - 1) = 0\\ \Longrightarrow\ &-4k^2 + 5 = 0\\ \Longrightarrow\ &k = \pm\tfrac{\sqrt{5}}{2}.\end{aligned}$$

— Alternative Method —

If the tangent has gradient $\frac{1}{2}$, then the radius must have gradient -2. The equation of the diameter is therefore $y = -2x$. Solving with $x^2 + y^2 = 1$ gives

$$x = \pm\tfrac{1}{\sqrt{5}}, \quad y = \mp\tfrac{2}{\sqrt{5}}.$$

Substituting into $y = \frac{1}{2}x + k$ yields $k = \pm\frac{\sqrt{5}}{2}$.

2360. The numbers of dots on opposite faces of a die add up to 7. So, the four vertical faces of each die will show 14 dots, giving $14n$ dots. The minimum and maximum totals are then with either 1 or 6 dots showing on the horizontal face of the top die. This gives $14n + 1 \le N \le 14n + 6$, as required.

2361. At $x = \pi$, the sine graph is at $(\pi, 0)$. Its gradient is $\cos\pi = -1$. Hence, the tangent line is $y = -x + \pi$. The tangent line is the best linear approximation to sin at $x = \pi$, as required.

— Alternative Method —

The small-angle approximation for x in radians is $\sin x \approx x$. To transform this approximation, we reflect in $x = \pi/2$. This is the same as

- reflecting in $x = 0$, so $x \mapsto -x$, then
- translating by π, so $-x \mapsto \pi - x$.

So, the approximation is $\sin x \approx \pi - x$, as required.

2362. The conditions give

$$\begin{aligned}4.8 &= A \times 2^k,\\ 9.6 &= A \times 2^{2k}.\end{aligned}$$

Dividing the latter by the former yields $2 = 2^k$. Hence, $k = 1$ and $A = 2.4$. So, $P = 2.4 \cdot 2^t$. At $t = 7$, this gives $P = 307.2$. Hence, the predicted population is 307.2 million.

— Alternative Method —

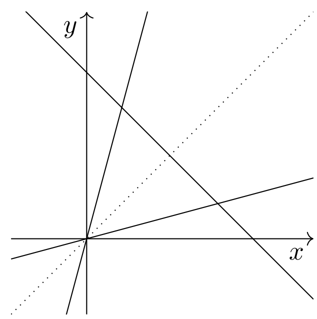
Exponential growth produces a constant doubling time (here 1 hour). After 7 hours, the population has doubled 6 times from 4.8 million. This gives a resulting population of

$$4.8 \times 2^6 = 307.2 \text{ million}.$$

2363. Writing this longhand, we can then simplify:

$$\begin{aligned}&\frac{x}{x} + \frac{x+1}{x-1} + \frac{x+2}{x-2} = 0\\ \Longrightarrow\ &(x-1)(x-2) + (x+1)(x-2)\\ &\qquad + (x+2)(x-1) = 0\\ \Longrightarrow\ &3x^2 - 3x - 2 = 0\\ \Longrightarrow\ &x = \frac{3 \pm \sqrt{33}}{6}.\end{aligned}$$

2364. The lines $y = \left(2 \pm \sqrt{3}\right)x$ meet at the origin. Also $\left(2 + \sqrt{3}\right)\left(2 - \sqrt{3}\right) = 1$, so the lines are reflections in $y = x$. The line $x + y = k$ is also symmetrical in $y = x$.



The angle between $y = \left(2 - \sqrt{3}\right)x$ and the x axis is $\arctan\left(2 - \sqrt{3}\right) = 30°$, which makes the angle at the origin $60°$. By symmetry, then, the lines form an equilateral triangle, which is the required result.

2365. One piece can be placed without loss of generality. Then, in order for the pattern to have rotational symmetry order 4, the orientations of the other three pieces are fixed by the first one: each has a probability of $1/4$ of maintaining symmetry. Hence, the probability is $1/4^3 = 1/64$.

2366. Equating the instances of the common ratio,

$$\frac{2c}{c-1} = \frac{5c+3}{2c}$$
$$\Longrightarrow c = -1, 3.$$

The common ratio is then $2c/c-1 = 1, 3$, which gives the first term as -2 or $2/3$.

2367. (a) The derivative of $\sec x$ is $\sec x \tan x$, which is $\sin x \sec^2 x$. For $x \in (-\pi/2, 0)$, the value of $\sin x$ is negative. Hence, since $\sec^2 x \geq 0$, the gradient is negative. Therefore, the curve is decreasing over this domain.

(b) The second Pythagorean trig identity gives $\tan x = \pm\sqrt{\sec^2 x - 1}$. So, if $y = \sec x$, then

$$\begin{aligned}\frac{dy}{dx} &= \sec x \tan x \\ &= \pm y\sqrt{y^2 - 1}.\end{aligned}$$

The choice of sign depends on the domain. For $(-\pi/2, 0)$, $y > 0$ but, as shown in (a), $\frac{dy}{dx} < 0$. Hence, we need the negative square root. This gives $\frac{dy}{dx} = -y\sqrt{y^2-1}$, as required.

2368. Let $u = 1 + 2x$. Then $du = 2dx$, so $dx = \frac{1}{2}du$. Also $x = \frac{1}{2}(u-1)$. This gives

$$\begin{aligned}&\int_0^4 \frac{3x}{\sqrt{1+2x}}\,dx \\ =& \int_1^9 \frac{3 \cdot \frac{1}{2}(u-1)}{\sqrt{u}}\,\frac{1}{2}du.\end{aligned}$$

Taking out a factor of $\frac{3}{4}$ and splitting the fraction up, this is

$$\begin{aligned}&\tfrac{3}{4}\int_1^9 u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,du \\ =& \tfrac{3}{4}\left[\tfrac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^9 \\ =& \tfrac{3}{4}\big((18-6) - (2/3 - 2)\big) \\ =& 10, \text{ as required.}\end{aligned}$$

2369. For stationary points of $y = \cos^2 x + \cos x$,

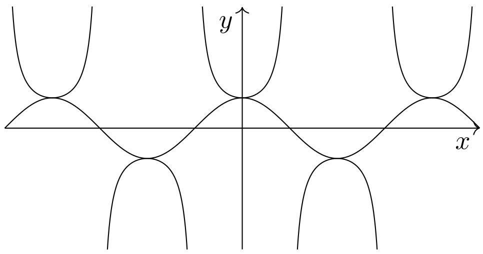
$$\begin{aligned}&-2\cos x \sin x - \sin x = 0 \\ \Longrightarrow\ & \sin x(2\cos x + 1) = 0 \\ \Longrightarrow\ & \sin x = 0 \text{ or } \cos x = -\tfrac{1}{2}.\end{aligned}$$

In the relevant domain, this gives SPs at $(0, 2)$, $(2\pi/3, -1/4)$, $(\pi, 0)$ and $(4\pi/3, -1/4)$. Hence, the range of $x \mapsto \cos^2 x + \cos x$ is $[-1/4, 2]$.

When reciprocated, an interval (a, b) containing zero gives two intervals to $\pm\infty$. So, the range of f is $(-\infty, -4] \cup [1/2, \infty)$.

Nota Bene

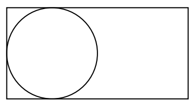
The effect of reciprocating an interval may be seen in the graphs of $y = \cos x$ and $y = \sec x$:



When reciprocated, the range $[-1, 1]$ of $\cos x$ is the range $(-\infty, -1] \cup [1, \infty)$ of $\sec x$.

2370. (a) Yes. The circumcentre of a circle is exactly the centre of such a circle.

(b) No. A 2×1 rectangle is a counterexample.



(c) Yes. Place a circle at the centre the n-gon, and enlarge it. Symmetry dictates it will touch all of the sides simultaneously.

2371. It reduces the probability to zero. The outcomes which sum to 7 are $(1, 6), (2, 5), (3, 4)$ and vice versa. In none of these is one score double the other, so $X = 2Y$ rules $X + Y = 7$ out.

2372. Being defined as x and y coordinates on the unit circle, $\cos\theta$ and $\sin\theta$ are reflections in $\theta = \frac{\pi}{4}$, i.e. $\sin\theta \equiv \cos\left(\frac{\pi}{2} - \theta\right)$, and vice versa.

(a) $\tan\left(\frac{\pi}{2} - \theta\right) \equiv \dfrac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos\left(\frac{\pi}{2} - \theta\right)} \equiv \dfrac{\cos\theta}{\sin\theta} \equiv \cot\theta.$

(b) This is given by the same symmetry: $\frac{\pi}{4} + \theta$ and $\frac{\pi}{4} - \theta$ are reflections of each other in $\theta = \frac{\pi}{4}$.

2373. (a) Applying the function twice,

$$\begin{aligned}f^2(x) &= \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} \\ &\equiv \frac{x+1+x-1}{x+1-(x-1)} \\ &\equiv \frac{2x}{2} \\ &\equiv x.\end{aligned}$$

Hence, applying the function twice returns the input. This means f is self-inverse.

(b) We require $\mathrm{f}^2(x) \equiv x$. This is

$$\begin{aligned} \frac{\frac{x+a}{x+b}+a}{\frac{x+a}{x+b}+b} &\equiv x \\ \Longrightarrow \frac{x+a+a(x+b)}{x+a+b(x+b)} &\equiv x \\ \Longrightarrow (a+1)x+a+ab & \\ &\equiv (b+1)x^2+\big(a+b^2\big)x. \end{aligned}$$

The coefficient of x^2 requires that $b+1=0$, so $b=-1$. This gives $(a+1)x \equiv (a+1)x$, which is automatically true. Hence, $b=-1$, but a can take any value.

2374. There is a common factor of $(x^2+1)^4$:

$$\begin{aligned} &(x^2+1)^5+(x^2+1)^4(x-7)=0 \\ \Longrightarrow\ &(x^2+1)^4(x^2+1+x-7)=0 \\ \Longrightarrow\ &(x^2+1)^4(x+3)(x-2)=0. \end{aligned}$$

The quadratic factor has no real roots, which gives the solution $x \in \{-3, 2\}$.

2375. Assume that the lower block doesn't slip, and the upper block is on the point of slipping. Then the frictional force on the upper block is $F_{\max}=\mu_1 mg$. Resolving along the string for the two connected blocks, $mg-\mu_1 mg=0$, so $\mu_1=1$.

Now, assume that the blocks don't slip against one another, and that the combined $2m$ block is on the point of slipping. So $F_{\max}=2\mu_2 mg$. Resolving along the string for the whole system, $mg-2\mu_2 mg=0$, so $\mu_2={}^1\!/_2$.

For equilibrium, $\mu_1 \in [1,\infty)$ and $\mu_2 \in [{}^1\!/_2,\infty)$.

2376. Factorising top and bottom,

$$\begin{aligned} &\frac{xz+yz-x^2-xy}{2xz+4yz-2x^2-4xy} \\ \equiv\ &\frac{(x+y)(z-x)}{2x+4y)(z-x)} \\ \equiv\ &\frac{x+y}{2x+4y)}. \end{aligned}$$

2377. The area formula gives $x=\frac{1}{2}r^2\cdot 2x=r^2x$. The arc length tells us that $x\neq 0$, so $1=r^2$, giving the radius as 1. The arc length formula then gives $(x+1)=r\cdot 2x$, so $x=1$.

2378. (a) The normal is $y=-2x$.

(b) They intersect at $(2,-4)$.

(c) Q is squared distance from the origin. Hence, we need the shortest distance from C to the origin. This shortest distance must lie along the normal. So we calculate Q at point $(2,-4)$. This gives $Q_{\min}=2^2+(-4)^2=20$.

2379. Squaring both sides and rearranging,

$$\begin{aligned} &\sqrt{6x-9}+\sqrt{2x-5}=x-1 \\ \Longrightarrow\ &6x-9+2\sqrt{(6x-9)(2x-5)}+2x-5 \\ &\qquad =x^2-2x+1 \\ \Longrightarrow\ &2\sqrt{12x^2-48x+45}=x^2-10x+15. \end{aligned}$$

Squaring both sides again,

$$\begin{aligned} &48x^2-192x+180=(x^2-10x+15)^2 \\ \Longrightarrow\ &x^4-20x^3+82x^2-108x+45=0. \end{aligned}$$

This can be solved using a calculator's polynomial solver, or by a numerical method, or by spotting roots and factorising. The roots are $x=1,3,15$.

But squaring twice has generated phantom roots. Substituting back in, only $x=15$ is a root of the original equation.

2380. The cubic equation $ax^3-bx=0$ can be factorised to $x(ax^2-b)=0$, giving $x=0$ or $ax^2-b=0$. Since $a,b>0$, this quadratic has two distinct real roots. So, $y=ax^3-bx$ would cross the x axis three times. But the graph shown only crosses once.

2381. We don't need to calculate p and q. The new roots are reflections of the old in $x=0$. We can enact this reflection by replacing x by $-x$. So, the new equation is $3(-x)^2+2(-x)-14=0$, which we can simplify to $3x^2-2x-14=0$.

2382. Both are true, for the same reason. Statement (b) is true because $\mathrm{g}(x)$ can take any output value in $[0,1]$, which is scaled by output transformations to the set $[b,a+b]$. Statement (a) is true because it is implied by (b).

2383. Enacting the differential operator,

$$\begin{aligned} &\frac{d}{dx}(x^2-2y)=1 \\ \Longrightarrow\ &2x-2\frac{dy}{dx}=1 \\ \Longrightarrow\ &\frac{dy}{dx}=x-\tfrac{1}{2}. \end{aligned}$$

Reciprocating this,

$$\frac{dx}{dy}=\frac{1}{x-\frac{1}{2}}\equiv\frac{2}{2x-1}.$$

2384. Integrating the jerk, $a=-120t+c$. At $t=0$, we know that $a=0$, because velocity was constant up to $t=0$. Hence $a=-120t$. To find the reduction in speed, we integrate this definitely:

$$\Delta v=\int_0^{0.5}-120t\,dt=\Big[-60t^2\Big]_0^{0.5}=-15.$$

Hence, the total reduction is 15 ms^{-1}.

— Nota Bene —

Each integration step could be performed either indefinitely or definitely. But the second integral from a to v is more naturally performed definitely, as it is a calculation of a *change* in velocity. A definite integral bypasses any consideration of the speed prior to the collision.

2385. (a) Setting $y = ax + b$, we get $x = \frac{1}{a}(y - b)$, so $\mathrm{f}^{-1}(x) = \frac{1}{a}(x - b)$. So, for f to be self-inverse, we need

$$ax + b \equiv \tfrac{1}{a}(x - b).$$

Equating coefficients of x gives $a = \frac{1}{a}$, so $a = \pm 1$. The constant terms then give $b = -\frac{b}{a}$. So, if $a = 1$, then $b = 0$; if $a = -1$, then b can be any real number.

(b) The lines in question are $y = x$ and $y = -x + b$. These are precisely all of the straight lines which are symmetrical in $y = x$.

2386. (a) The distance between centres is 4, which is also r. So, intersections A, B are at the vertices of equilateral triangles, side length 4, above and below the x axis. These are $\left(2, \pm 2\sqrt{3}\right)$.

(b) The radii to the intersections A, B subtend $\frac{2\pi}{3}$ at O. Sector OAB has area

$$\tfrac{1}{2} \cdot 4^2 \cdot \tfrac{2\pi}{3} = \tfrac{16\pi}{3}.$$

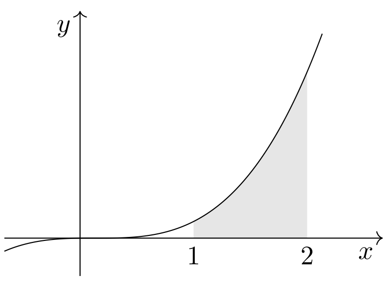
The area of triangle OAB is then

$$\tfrac{1}{2} \cdot 4^2 \cdot \sin \tfrac{2\pi}{3} = 4\sqrt{3}.$$

So, the segment has area $\frac{16\pi}{3} - 4\sqrt{3}$. To get the area common to both circles, we double to $\frac{32\pi}{3} - 8\sqrt{3}$, as required.

2387. There are only two successful outcomes, which are HTHTHT and THTHTH. The possibility space has $2^6 = 64$ total outcomes, so $p = \frac{\text{successful}}{\text{total}} = \frac{1}{32}$.

2388. (a) The positive cubic $y = (3x - 1)^3$ has a triple root at $x = \frac{1}{3}$. The area in question is to the right of this:



(b) The correct calculation is

$$\int_1^2 (3x - 1)^3 \, dx = \left[\tfrac{1}{3} \cdot \tfrac{1}{4}(3x - 1)^4\right]_1^2 = 50.75.$$

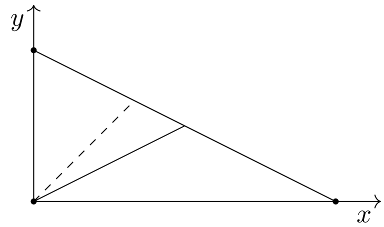
The student's calculation of 152.25 is three times bigger than this, suggesting that the area scale factor of $\frac{1}{3}$, which emerges by the reverse chain rule, has been forgotten.

2389. On a clock, degrees are easier: $\frac{3\pi}{10}$ radians is 54°. One minute is 6°; one hour is 30°. At m minutes before 8pm, the minute hand is $6m°$ from vertical. The hour hand is $\frac{m}{60} \times 30° = \frac{1}{2}m$ from 8pm, so $120 + \frac{m}{60} \times 30°$ from vertical. So, we set

$$\begin{aligned} &120 + \tfrac{1}{2}m - 6m = 54 \\ \Longrightarrow\ & m = 12. \end{aligned}$$

The time is 7:48pm.

2390. Consider a triangle with vertices at $(0, 0)$, $(2, 0)$ and $(0, 1)$.



The median from $(0, 0)$ (solid) is $y = \frac{1}{2}x$, but the angle bisector (dashed) is $y = x$. This disproves the claim.

2391. Let $\mathrm{q}_1(x)$ and $\mathrm{q}_2(x)$ be $\mathrm{f}_1(x)/\mathrm{g}_1(x)$ and $\mathrm{f}_2(x)/\mathrm{g}_2(x)$, where $\mathrm{f}_1, \mathrm{f}_2, \mathrm{g}_1, \mathrm{g}_2$ are polynomial functions. Then the sum is

$$\begin{aligned} \mathrm{q}_1(x) + \mathrm{q}_2(x) &= \frac{\mathrm{f}_1(x)}{\mathrm{g}_1(x)} + \frac{\mathrm{f}_2(x)}{\mathrm{g}_2(x)} \\ &\equiv \frac{\mathrm{f}_1(x)\,\mathrm{g}_2(x) + \mathrm{f}_2(x)\,\mathrm{g}_1(x)}{\mathrm{g}_1(x)\,\mathrm{g}_2(x)}. \end{aligned}$$

A product/sum of polynomials is polynomial, so numerator and denominator are both polynomial. Hence, the sum $x \longmapsto \mathrm{q}_1(x) + \mathrm{q}_2(x)$ is a rational function. □

2392. (a) We need the following identity to hold:

$$\begin{aligned} &3x^3 - 25x^2 + 56x - 16 \\ &\equiv 3x(x - k)^2 - (x - k)^2 \\ &\equiv 3x\big(x - 2kx + k^2\big) - \big(x^2 - 2kx + k^2\big). \end{aligned}$$

The x^3 terms match. Equating coefficients of x^2, we require $-25 = -6k - 1$, so $k = 4$.

(b) Taking out $(x - 4)^2$,

$$\begin{aligned} &(x - 4)^2(3x - 1) = 0 \\ \Longrightarrow\ & x = 4, \tfrac{1}{3}. \end{aligned}$$

2393. The vector is $a\mathbf{i} + a\mathbf{j} + (2a - 1)\mathbf{k}$. By Pythagoras, the squared length is $a^2 + a^2 + (2a - 1)^2 \equiv 6a^2 - 4a + 1$. Equating this to 81, $a = -10/3, 4$.

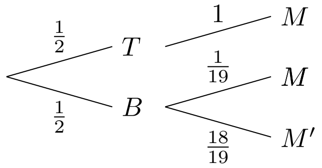
2394. (a) The graph has a vertical asymptote where the denominator is zero. This is $x = -2$.

(b) Assuming that the line is $y = k$ and tangent to the curve, we seek points with zero gradient. The quotient rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{(e^x)'(x+2) - e^x(x+2)'}{(x+2)^2} \\ &\equiv \frac{xe^x + e^x}{(x+2)^2} \\ &\equiv \frac{e^x(x+1)}{(x+2)^2}.\end{aligned}$$

Since $e^x \neq 0$, this is zero if and only if $x = -1$. Substituting back in, the line is $y = {}^1\!/_e$.

2395. The tree diagram, conditioned on choice of drawer, is as follows:



(a) $\mathbb{P}(\text{match}) = \frac{1}{2} + \frac{1}{2} \times \frac{1}{19} = \frac{10}{19}$.

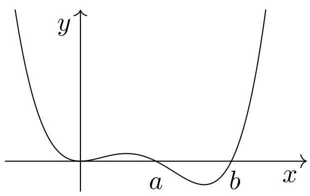
(b) Restricting the possibility space,

$$\begin{aligned}&\mathbb{P}(\text{bundled} \mid \text{match}) \\ &= \frac{\mathbb{P}(\text{bundled} \cap \text{match})}{\mathbb{P}(\text{match})} \\ &= \tfrac{19}{20}.\end{aligned}$$

2396. We set up $a(k+1)^3 + b(k+1)^2 + c(k+1) + d$ and equate coefficients, beginning with k^3. This requires $a = 1$, giving $k^3 + 3k^2 + 3k + 1$. So, $b = 0$ and $c = 1$. Then $d = 10$. So,

$$k^3 + 3k^2 + 4k + 12 \equiv (k+1)^3 + (k+1) + 10.$$

2397. This is a positive quartic. It has a double root (point of tangency) at $x = 0$, and single roots at $x = a, b$:



2398. For positive x, these obviously hold. For negative x, the even powers eliminate the minus sign, while the odd power retains it, making $\sqrt[3]{x^3} < 0$ negative when $x < 0$.

(a) True,

(b) False,

(c) True.

2399. Let $X = 2x + 5$, and the graph is $y = X^4 - X^2 + 2$. This is a positive quartic, and, since it has no odd-powered terms, the y axis is a line of symmetry. The W shape of a quartic dictates, therefore, that the local maximum must be at $X = 0$. Converting back to x, this gives $(-{}^5\!/_2, 2)$.

2400. (a) The factor of 3 can be taken out of the sum, giving $\frac{\pi^2}{2}$.

(b) This sum is missing the $r = 1$ term, whose value is 1. So, the sum is $\frac{\pi^2}{6} - 1$.

(c) Firstly, a factor of $\frac{1}{2}^2$ can be taken out. This leaves the denominator as $(r+1)$, which is the same as starting the sum from $r = 2$ with a denominator of r. Hence, the answer is $\frac{1}{4}$ that of (b): $\frac{\pi^2}{24} - \frac{1}{4}$.

—— End of 24th Hundred ——